Riemann Surfaces (Milnor, Ch. 1)

1-dimensional complex manifold
(holomorphic change of coordinates)

Thm (Uniformization theorem)

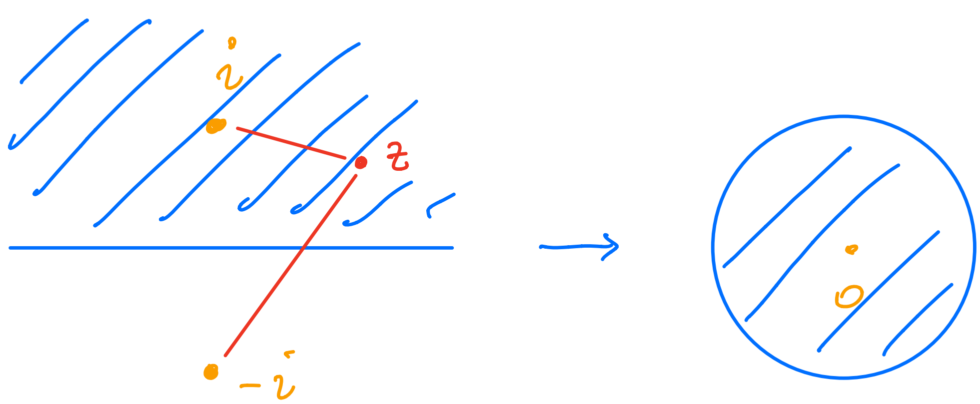
Every simply connected Riemann surface is isomorphic to either:

1. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ — Riemann sphere, SPHERICAL (positive curv.)
2. $\mathbb{C}$ — plane, EUCLIDEAN / FLAT $(K=0)$
3. $\mathbb{D}$ — (Poincaré) disc, HYPERBOLIC $(K<0)$

   is

   $\mathbb{H}$ — upper half plane

$$z \longmapsto \frac{z-i}{z+i}$$



$$\left|\frac{z-i}{z+i}\right| < 1 \quad \longleftrightarrow \quad |z-i| < |z+i|$$

Cor.: every R.S. is obtained by a quotient of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$ by a discrete group of automorphisms.

Poincaré Disk

$\mathrm{Aut}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{D}$ holo with holo inverse$\}$

Lemma (Schwarz)

If $f : \mathbb{D} \to \mathbb{D}$ holo with $f(0) = 0$, then $|f'(0)| \leq 1$ with equality iff f is a rotation.

Pf.: $g(z) = \dfrac{f(z)}{z}$ well defined $g : \mathbb{D} \to \mathbb{C}$

apply max. principle. $(r<1)$

$$\sup_{\mathbb{D}_r} |g(z)| = \sup_{\partial \mathbb{D}_r} |g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r} \xrightarrow{r \to 1} 1$$

$$|g(z)| = \frac{|f(z)|}{|z|} \leq 1$$

$$|f(z)| \le |z| \quad \forall z \in \mathbb{D}.$$

$$\lim_{z \to 0} \frac{f(z)}{z} = f'(0) \implies |f'(0)| \le 1.$$

if $|g(z_0)| = 1 \implies g(z) = c = \frac{f(z)}{z}$

$$f(z) = cz$$

Cor.: if $f(\mathbb{D}_r) \subset \mathbb{D}_s$ for r, s, $f(0)=0$ and f holo, then

$$|f'(0)| \le \frac{s}{r}$$

Cor.: take $r \to \infty$

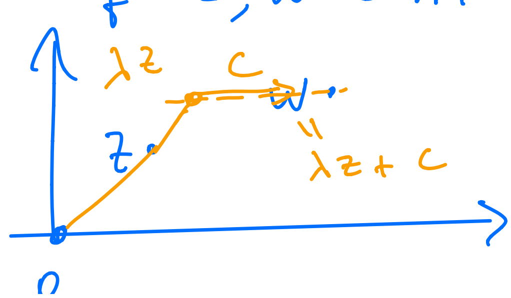
if $f(\mathbb{C}) \subset \mathbb{D}_s$, then $|f'(0)| = 0$

so f is constant

$\implies$ Liouville's theorem

Lemma Given $z, w \in \mathbb{D}$, there exists a $g \in \mathrm{Aut}(\mathbb{D})$ s.t. $g(z) = w$.

Pf if $z, w \in \mathbb{H}$



$\mathrm{Aut}(\mathbb{H})$
$\cup$
$\{z \mapsto \lambda z + c,\ \lambda > 0,\ c \in \mathbb{R}\}$

Thm $\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \dfrac{az+b}{cz+d} \;/\; \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc>0 \end{array} \right\}$

Pf $g \in \mathrm{Aut}(\mathbb{H})$, $i \in \mathbb{H}$

$z_0 = g(i)$

① $\exists\, g' \in PSL_2(\mathbb{R})$ s.t. $g'(z_0) = i$

② $h = g' \circ g \in \mathrm{Aut}(\mathbb{H})$

$h(i) = i$

$$\begin{array}{ccc} i \in \mathbb{H} & \xrightarrow{h} & \mathbb{H} \ni i \\ \varphi \downarrow & & \downarrow \varphi \\ 0 \in \mathbb{D} & \longrightarrow & \mathbb{D} \ni 0 \end{array}$$

$$\left.\begin{array}{l} h' = \varphi \circ h \circ \varphi^{-1} \in \mathrm{Aut}\,\mathbb{D} \\ h'(0) = 0 \end{array}\right\} \longrightarrow h \text{ is rotation}$$

$$\mathbb{D} \xrightarrow{h} \mathbb{D} \xrightarrow{h^{-1}} \mathbb{D}$$

$$|h'(0) \cdot (h^{-1})'(0)| = 1$$

$\Longrightarrow |h'(0)| = 1 \Longrightarrow h$ is rotation

③ $h = g' \circ g \in PSL_2(\mathbb{R})$

$$\Longrightarrow \quad g = (g')^{-1} \circ h \in PSL_2(\mathbb{R})$$

$$PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\} \Big/ \{\pm I\}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}(z) = \frac{-z}{-1} = z$$

$$\mathrm{Aut}(\mathbb{H}) \simeq PSL_2(\mathbb{R})$$

Metric on $\mathbb{D}$ or $\mathbb{H}$

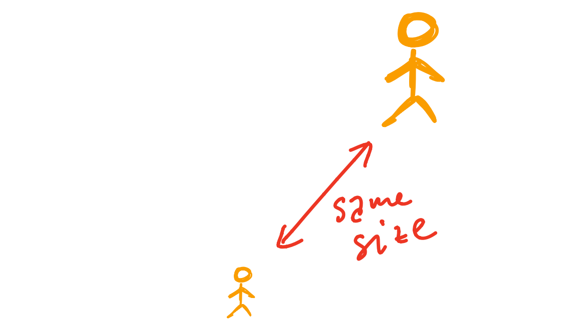
Thm There exists a <u>unique</u>, up to constant, Riemannian metric on $\mathbb{H}$ (or on $\mathbb{D}$) which is invariant under all automorphism of $\mathbb{H}$ (or $\mathbb{D}$).

It is called the HYPERBOLIC or POINCARÉ metric.

$\mathbb{H}$ $\qquad ds = \dfrac{dz}{y}$ $\qquad$ euclidean

$$\mathbb{H} = \{x + iy,\ y > 0,\ x \in \mathbb{R}\}$$

Rmk This metric is CONFORMAL, i.e.
it is of form $ds = \rho(z)\, dz$
for $\rho : \mathbb{H} \to \mathbb{R}^{+}$



$g(z) = \lambda z + c \quad , \quad \lambda > 0, \quad c \in \mathbb{R}$

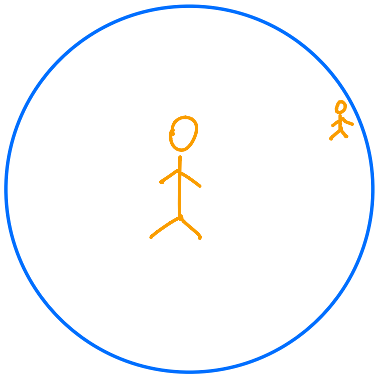
$w = \lambda z + c \qquad z = x + iy$

$dw = \lambda\, dz \qquad w = \lambda x + \lambda i y$

$\qquad Im(w) = \lambda\, Im(z)$

$$\frac{dz}{y} = \frac{dz}{Im(z)} = \frac{\lambda\, dz}{\lambda\, Im(z)} = \frac{dw}{Im(w)}$$

$\mathbb{D}$ $\quad ds = \dfrac{2 \cdot dz}{1 - |z|^2}$



<u>Heuristic</u>: $\quad ds_u \approx \dfrac{dz}{d(z, \partial U)}$

<u>Rmk</u> with this definition, the Gaussian curvature is $K \equiv -1$ (if you remove 2, then $K \equiv -4$)